Determination of Geopotential Coefficients by Efficient Numerical Integration Techniques

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An efficient, high-accuracy, double-integration method that evaluates the iterated form of the surface integral representations of the geopotential coefficients is presented. The efficiency and high accuracy of this method is obtained by the utilization of the Romberg numerical integration scheme. Additional efficiency is obtained by computing the inner integral common to many different coefficients only once. This method was tested using a test case earth that represented the basic features of the earth's potential field.

I. Introduction

THE efficient and accurate reduction of global data sets to a spherical harmonic representation of the earth's potential field has been one of the main focuses of geodesy (see, e.g., Refs. 1 and 2). This reduction is normally accomplished by using least squares or the orthogonality properties of the spherical harmonic functions (see, e.g., Ref 3 or 4). The report by Colombo[4] introduces the use of a fast-Fourier algorithm to increase the efficiency of computing higher-order models. This paper introduces the use of the Romberg numerical integration scheme to increase the efficiency of computing spherical harmonic representations of the earth's potential field while maintaining or increasing accuracy. Unlike the fast-Fourier algorithm that may be inefficient when only a few coefficients are computed, the Romberg procedure is uniformly more efficient.

This presentation will not address the problems associated with generating an accurate gravimetric database for a sphere, but rather will assume that the database for the utilization of numerical integration techniques is known. Thus, the main focus of this paper will be the theoretical foundation for recovering the geopotential coefficients from gravimetric data on a sphere, the determination of the accuracy of these recovered coefficients, and the cost of implementation.

II. Theoretical Foundation

Since the preliminary theoretical development is found in Ref. 3, it will not be repeated here. Using the notation of Heiskanen and Mortiz,[3] we denote the potential at latitude θ and longitude λ on the sphere σ that contains the earth by $f(\theta,\lambda)$. The Legendre and associated Legendre functions are $P_n(x)$ and $P_{nm}(x)$, respectively, and the unnormalized geopotential coefficients are given by the equation set

$$a_{n0} = \frac{2n+1}{4\pi}\int_\sigma\int f(\theta,\lambda)\,P(\cos\theta)\,d\sigma \tag{1}$$

$$a_{nm} = K\int_\sigma\int f(\theta,\lambda)\,P_{nm}(\cos\theta)\cos m\lambda\,d\sigma$$

$$a_{nm} = K\int_\sigma\int f(\theta,\lambda)\,P_{nm}(\cos\theta)\sin m\lambda\,d\sigma \qquad m\neq 0 \tag{2}$$

where

$$K = \frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}$$

Changing these surface integrals into iterated integrals where the differential $d\sigma = \sin\theta\,d\lambda\,d\theta$ and the latitude varies between 0, π, and the longitude varies between 0, 2π, then the geopotential coefficients are given by

$$a_{n0} = \frac{2n+1}{2\pi}\int_0^\pi P_n(\cos\theta)\sin\theta\,A_0(\theta)\,d\theta \tag{3}$$

$$a_{nm} = K\int_0^\pi P_{nm}(\cos\theta)\sin\theta\,A_m(\theta)\,d\theta$$

$$b_{nm} = K\int_0^\pi P_{nm}(\cos\theta)\sin\theta\,B_m(\theta)\,d\theta \qquad m\neq 0 \tag{4}$$

where

$$A_m(\theta) = \int_0^{2\pi} f(\theta,\lambda)\cos m\lambda\,d\lambda \qquad m=0,1,,\ldots,n \tag{5}$$

$$B_m(\theta) = \int_0^{2\pi} f(\theta,\lambda)\sin m\lambda\,d\lambda \qquad m=1,\ldots,n \tag{6}$$

Note that $A_m(\theta)$ and $B_m(\theta)$ need only be computed once for each fixed value of θ, and then they can be used in computing all the a_{nm}'s and b_{nm}'s (Colombo also exploits this structure to gain efficiency). Thus, the geopotential coefficients can be computed without using surface integral numerical techniques, but instead by the very efficient Romberg integration on both the inner and outer integrals.

III. Romberg Numerical Integration

This section will cover the major concepts that make the Romberg numerical integration so powerful. The integral of $f(x)$ from a to b

$$I = \int_a^b f(x)\,dx$$

is first approximated by the composite trapezoidal rule of N subdivisions, and hence, step size $h=(b-a)/N$. The composite trapezoidal approximation $T(h)$ is given by

$$T(h) = h\left(f_0/2 + \sum_{i=1}^{n-1} f_i + f_n/2\right) \tag{7}$$

where $x_i = a + ih$ and $f_i = f(x_i)$, $i = 0,1\ldots N$.

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If $f(x)$ is analytic (i.e., $f(x)$ can be expanded in an infinite Taylor's series), then $T(h)$ can be expressed as

$$T(h) = I + c_2h^2 + c_4h^4 + c_6h^6 + \cdots \tag{8}$$

where $c_2, c_4, c_6, \cdots$ are independent of h (see, e.g., Ref. 5). This representation of $T(h)$ shows not only the error in the approximation but also its error structure. Since for small h, the most significant contributor to this error is c_2h^2, the error term is said to be of order two. By extrapolation this error structure is exploited to obtain a more accurate approximation by eliminating the error term of order two.

If the number of subdivisions is doubled, then the composite trapezoidal rule $T(h/2)$ would have the error structure

$$T(h/2) = I + c_2h^2/4 + c_4h^4/16 + c_6h^6/64 + \cdots \tag{9}$$

and

$$4T(h/2) = 4I + c_2h^2 + c_4h^4/4 + c_6h^6/16 + \cdots \tag{10}$$

Note that the second-order terms of the error in Eqs. (8) and (10) are now identical. Subtracting Eq. (8) from (10) and dividing by 3, we obtain an extrapolated approximation

$$T^1(h) = [4T(h/2) - T(h)]/3$$

This new approximation can be expressed

$$T^1(h) = I + D_4h^4 + D_6h^6 + \cdots \tag{11}$$

where

$$D_4 = -c_4/4, \quad D_6 = -5c_6/16, \cdots$$

Hence, $T^1(h)$ has an error term of order four. Note this is two orders better than $T(h)$.

Now starting with $T^1(h)$ and $T^1(h/2)$, the aforementioned process can be repeated and the approximation $T^2(h)$ calculated

$$T^2(h) = [16\,T^1(h/2) - T^1(h)]/15$$

The $T^2(h)$ approximation has the error structure

$$T^2(h) = I + E_6h^6 + \cdots \tag{12}$$

and again, its error term is two orders better than $T^1(h)$.

The extrapolation procedure used in Romberg integration requires the computation of the sequence

$$T(h), T(h/2), T(h/4), T(h/8), \ldots \tag{13}$$

to produce the sequence of extrapolations

$$T^1(h), T^2(h), T^3(h), T^4(h), \ldots \tag{14}$$

Theoretically, if the sequence (13) converges to I, then the sequence (14) converges to I and its rate of convergence is much faster (see, e.g., Ref. 6). However, since computers do not have infinite word length, the error in the application of the Romberg integration does not approach zero as the step size h approaches zero. Rather, the error will first decrease rapidly as the step h gets small, then the error will level off for a short period and this error will rapidly increase as h decreases. This phenomenon occurs because the total error is the sum of the theoretical error (which approaches zero as h approaches zero) and the round-off error (which is initially small but approaches infinity as h approaches zero).

IV. Test Case Investigation

This section focuses on the accuracy of the preceding numerical techniques and utilizes a test case mode of investigation. The steps used in the test case mode of investigation are the following:

1) Select a test case that closely approximates the physical phenomena being modeled and for which the exact results are known.

2) Approximate the test case results, using exactly the same modeling procedures followed in modeling the real-world physical phenomena.

3) Measure the accuracy of the modeling procedures by determining the difference between parameters generated approximately and exactly.

Since NASA's Goddard Earth Model 10B (GEM10B) closely approximates the actual earth, it was selected as the test case earth (see, e.g., Ref. 7). Potentials were computed for points on the sphere $R = a$, where a is the semi-major axis of the earth; hence, the classical spherical harmonic representation of the potential at (θ, λ) simplifies

$$U = \frac{GM}{R}\left\{1 + \sum_{n=2}^{36}\sum_{m=0}^{n}\left[(a_{nm}\cos m\lambda + b_{nm}\sin m\lambda) \times P_{nm}(\cos\theta)\right]\right\}$$

where

G = universal gravitational constant.

M = mass of the earth.

R = semi-major axis of the earth.

a_{nm}, b_{nm} = the geopotential coefficients.

The data network used in this investigation was the set of points (θ_i, λ_j) such that

$$\theta_{i+1} = \theta_i + \pi/256, \qquad \theta_0 = 0$$
$$\lambda_{j+1} = \lambda_j + 2\pi/256, \qquad \lambda_0 = 0$$
$$i, j = 0,1,\ldots,255$$

The potential at each of these data points was computed using GEM10B and stored in a file P256 × 256.

The following subsets of this file were generated by deleting data from P256 × 256:

1) File P128 × 256 has points (θ_i, λ_j) such that

$$\theta_{i+1} = \theta_i + \pi/128, \qquad \theta = 0$$
$$\lambda_{j+1} = \lambda_j + 2\pi/256, \qquad \lambda_0 = 0$$
$$i = 0,1,\ldots,127$$
$$j = 0,1,\ldots,255$$

2) File P128 × 128 has points (θ_i, λ_j) such that

$$\theta_{i+1} = \theta_i + \pi/128, \qquad \theta_0 = 0$$
$$\lambda_{j+1} = \lambda_j + 2\pi/128, \qquad \lambda_0 = 0$$
$$i, j = 0,1,\ldots,127$$

3) File P64 × 128 has points (θ_i, λ_j) such that

$$\theta_{i+1} = \theta_i + \pi/64, \qquad \theta_0 = 0$$
$$\lambda_{j+1} = \lambda_j + 2\pi/128, \qquad \lambda_0 = 0$$
$$i = 0,1,\ldots,63$$
$$j = \theta,1,\ldots,127$$

Using the files P64 × 128, P128 × 128, P128 × 256, and P256 × 256 and the Romberg integration scheme on the inner and outer integrals (5), (6) and (3), (4), respectively, the geopotential coefficient through degree and order eight were computed and store in files M64 × 128, M128 × 128, M128 × 256, M256 × 256.

A. Accuracy of Coefficients

The accuracy of the coefficients stored in M64 × 128, M128 × 128, M128 × 256 and M256 × 256 were tested by comparing them with their corresponding coefficients from GEM10B. Table 1 lists the differences for the a_{n0}, a_{n1}, and a_{nn} $n = 2,3,\ldots,8$ coefficients. Note that the coefficients a_{60} and a_{80} are not as accurate as the a_{61} and a_{81} coefficients in model 64 × 128. This is due to the phenomenon called aliasing.

Table 1 Model accuracy errors in the a_{n0}, a_{n1}, and a_{nn} coefficients. Difference between GEM10B and Romberg 8 × 8 models

Coefficient a_{nm}	Data points 64 × 128	128 × 128	128 × 256	256 × 256
a_{20}	−6.1E-09	−8.8E-09	−6.0E-11	−2.6E-10
a_{30}	−3.6E-09	−3.2E-09	−1.5E-10	1.2E-12
a_{40}	−5.8E-08	5.9E-09	1.5E-09	2.0E-11
a_{50}	9.8E-09	4.6E-09	4.4E-10	1.2E-12
a_{60}	−1.1E-05	2.3E-09	2.0E-09	−1.3E-11
a_{70}	7.2E-09	−7.1E-09	−1.1E-09	2.0E-12
a_{80}	−1.5E-05	8.6E-09	8.6E-09	−9.3E-12
a_{21}	5.1E-10	9.9E-10	−1.4E-12	−1.2E-13
a_{31}	1.1E-09	9.1E-10	1.2E-11	1.1E-12
a_{41}	1.2E-09	−1.3E-09	−2.7E-12	−3.4E-13
a_{51}	−4.6E-09	−2.7E-09	1.8E-11	9.7E-13
a_{61}	−4.6E-11	1.1E-09	−3.7E-12	−3.4E-13
a_{71}	1.6E-09	3.0E-09	2.5E-11	1.4E-12
a_{81}	2.3E-09	−4.8E-10	−4.7E-12	−4.2E-13
a_{22}	−6.6E-09	−3.7E-09	2.4E-13	3.8E-13
a_{33}	−2.9E-10	−3.7E-09	2.6E-14	2.4E-14
a_{44}	1.2E-10	4.1E-11	−9.6E-16	−9.7E-16
a_{55}	−3.2E-12	−2.2E-12	9.2E-17	9.2E-17
a_{66}	−4.5E-13	−5.7E-13	2.7E-19	2.9E-19
a_{77}	1.6E-13	1.1E-13	−2.1E-20	−2.1E-20
a_{88}	3.4E-15	−1.2E-15	−3.8E-20	−3.8E-20

Aliasing occurs because of the coarse, equal-spaced sampling of a high-frequency term. This sampling procedure makes the high-frequency term appear as if it is of a lower frequency (see, e.g., Ref. 8). Thus, an increase in sampling frequency in the potential file should remove the problem of aliasing for this test case. Note, that in models 128 × 128, 128 × 256, and 256 × 256 where the sampling has increased by 2, 4, and 8 times, respectively, the problem of aliasing has disappeared. Table 1 also shows the general improvement in accuracy of each of the coefficients as the number of data points increases. However, the coefficients a_{22}, $a_{33},\ldots,a_{88}$ do not change much from model 128 × 256 to 256 × 256. Since there are only two error sources, the theoretical error and the round-off error, the near equality must result from either the theoretical error remaining constant as the step size in θ decreases or the increase in round-off balances the decrease in the theoretical error.

B. Accuracy of Computed Potentials

The ability of the models to reproduce the potential field on the network of points (θ_i, λ_j)

$$\theta_{i+1} = \theta_i + \pi/4, \qquad \theta_0 = 0; \quad i = 0,1,2,3$$

$$\lambda_{j+1} = \lambda_j + 2\pi/4, \qquad \lambda_0 = 0; \quad j = 0,1,2,3,4,5,6,7$$

was tested by comparing model potentials with GEM10B potentials. Table 2 lists the differences at each of the network points just presented. Note how much poorer model 64 × 128 is. Again, this is caused by the aliasing problem previously discussed.

Table 2 Model accuracy potential compared on 45 by 45° mesh Difference between GEM10B and Romberg 8 × 8 models

Latitude	Longitude	64 × 128	128 × 128	128 × 256	256 × 256
0	0	−17842.	1.605	6.8890	−0.2015
45	0	−1876.	−3.235	1.0762	−0.0401
45	45	−1941.	−16.135	1.2655	−0.0394
45	90	−1809.	−5.327	1.3646	−0.0391
45	135	−1930.	−2.785	1.3101	−0.0368
45	180	−1844.	3.596	1.1399	−0.0383
45	225	−1883.	−11.472	−0.9668	−0.0400
45	270	−1827.	10.490	0.8766	−0.0421
45	315	−1817.	0.125	0.9166	−0.0411
90	0	−364.	6.772	1.5293	0.0833
90	45	−385.	3.371	1.5495	0.0824
90	90	−330.	38.638	1.5732	0.0825
90	135	−374.	−1.207	0.5847	0.0870
90	180	−388.	8.569	1.5794	0.0863
90	225	−404.	−20.297	1.5715	0.0852
90	270	−359.	11.471	1.5545	0.0845
90	315	−394.	−15.438	1.5313	0.0828
135	0	−1879.	−7.393	0.9912	−0.0413
135	45	−1855.	1.570	0.8649	−0.0415
135	90	−1882.	3.782	0.8383	−0.0411
135	135	−1847.	2.253	0.9219	−0.0392
135	180	−1897.	−3.349	1.0810	−0.0365
135	225	−1849.	1.212	1.2278	−0.0362
135	270	−1851.	−7.973	1.2627	−0.0372
135	315	−1844.	7.438	1.1580	−0.0394

This data as well as the succeeding data were generated using a Sage 4 microcomputer that utilizes the Motorola 68,000 microprocessor chip. The following referenced computer runtimes for comparing different techniques were also determined on the Sage 4. The first comparison of techniques will be with the standard surface integral method and the second will be with the method of least squares.

C. Standard Surface Integral Technique

The geopotential coefficients were computed using surface integral techniques. The earth's surface was divided into 41,258 approximately equal area elements and the potentials at the midpoints of the elements were determined. Then, the geopotential coefficients were computed using Eqs. (1) and (2). The accuracy of this model's coefficients was tested by comparing them with the GEM10B coefficients. Table 3 lists the differences for the a_{nm}, $m = 0,1,2$, $n = 2,3,\ldots,8$ coefficients. For reference purposes, the results of the GEM10B and Romberg 128 × 256 coefficient comparison are also included.

The ability of the surface integral model to reproduce the potential field on the network described in Section IV.B was tested by comparing model potentials with GEM10B potentials; Table 4 lists the differences at each of the network points. Again for reference purposes, the results of the GEM10B and Romberg 128 × 256 model comparison on this network are also included.

Tables 3 and 4 clearly show that the 128 × 256 model is superior to the surface potential model. In addition, the surface potential model used more data points and required more computer runtime than the Romberg 128 × 256 model.

D. The Standard Least-Squares Method

The geopotential coefficients were computed using the method of least squares. The earth's surface was divided into 18,284 approximately equal area elements and the midpoint of

Table 3 Standard and Romberg integration comparisons Sage runtime, Standard = 48 min., Romberg = 34 min. Coefficient accuracy

a_{nm}	Standard 41,258 data points	Romberg 128 × 256 data points
a_{20}	−2.4E-07	−6.0E-11
a_{21}	−5.6E-11	−1.4E-12
a_{22}	7.4E-11	2.4E-13
a_{30}	−3.4E-07	−1.5E-10
a_{31}	1.2E-10	1.2E-11
a_{32}	−2.9E-11	2.4E-14
a_{40}	−3.9E-07	1.5E-09
a_{41}	−1.5E-10	−2.7E-12
a_{42}	−7.2E-11	3.2E-13
a_{50}	−6.6E-07	4.4E-10
a_{51}	5.4E-10	1.8E-11
a_{52}	5.4E-13	−5.0E-14
a_{60}	−2.4E-07	2.0E-09
a_{61}	4.3E-11	−3.7E-12
a_{62}	3.5E-12	4.5E-13
a_{70}	−1.0E-06	−1.1E-09
a_{71}	2.7E-10	2.5E-11
a_{72}	3.9E-11	−9.3E-14
a_{80}	−3.4E-07	−8.6E-09
a_{81}	−7.8E-11	−4.7E-12
a_{82}	5.2E-11	6.2E-13

Table 4 Standard and Romberg integration comparisons Sage runtime, Standard = 48 min, Romberg = 34 min. Potential accuracy

Latitude	Longitude	Standard 41,258 data points	Romberg 128 × 256 data points
0	0	−2161.13	6.8890
45	0	139.36	1.0762
45	45	141.57	1.2655
45	90	144.06	1.3646
45	135	140.61	1.3101
45	180	141.33	1.1399
45	225	142.64	0.9668
45	270	143.9	0.8766
45	315	140.75	0.9166
90	0	−29.99	1.5293
90	45	−29.49	1.5495
90	90	−30.00	1.5732
90	135	−29.97	1.5847
90	180	−30.14	1.5794
90	225	−30.16	1.5715
90	270	−29.51	1.5545
90	315	−30.10	1.5313
135	0	−97.88	0.9912
135	45	−97.40	0.8649
135	90	−96.16	0.8383
135	135	−95.91	0.9219
135	180	−97.10	1.0810
135	225	−95.74	1.2278
135	270	−95.98	1.2627
135	315	−96.91	1.1580
180	0	544.38	9.2066

each element was determined. The cost in computer runtime was 53 hours. The accuracy of the least-squares model coefficients were tested by comparing them with the GEM10B coefficients. Table 5 lists the differences for a_{nm}, $m = 0, 1, 2$, $n = 2, 3, \ldots, 8$ coefficients. For reference purposes, the results of the GEM10B and Romberg 256 × 256 coefficient comparison are also included.

The ability of the least-squares model to reproduce the potential field on the network described in Section IV.B was tested by comparing the least-squares model results with those of GEM10B. Table 6 lists the differences at each of the network points. The results of the GEM10B and Romberg 256 × 256 model comparison on the network are included for easy reference.

Table 5 Least-square and Romberg integration comparisons, Sage runtime least square = 53 hr, Romberg = 70 min. Coefficient accuracy

a_{nm}	Least squares 18,248 data points	Romberg 256 × 256 data points
a_{20}	2.5E-10	−2.6E-10
a_{21}	2.0E-11	−1.2E-13
a_{22}	−2.4E-11	3.8E-13
a_{30}	1.4E-10	1.2E-12
a_{31}	1.7E-11	1.1E-12
a_{32}	8.5E-12	7.2E-14
a_{40}	4.2E-10	−2.0E-11
a_{41}	4.3E-11	−3.4E-12
a_{42}	−2.2E-12	2.1E-14
a_{50}	1.9E-10	1.2E-12
a_{51}	2.3E-11	9.7E-13
a_{52}	5.03-12	2.3E-14
a_{60}	6.3E-10	−1.3E-11
a_{61}	3.6E-11	−3.4E-13
a_{62}	−1.2E-12	4.4E-15
a_{70}	2.3E-10	2.0E-12
a_{71}	6.3E-11	1.4E-12
a_{72}	5.8E-12	5.4E-15
a_{80}	8.9E-10	9.3E-12
a_{81}	3.5E-11	−4.2E-13
a_{82}	−6.4E-12	5.4E-15

Table 6 Least-squares and Romberg integration comparisons, Sage runtime least squares = 53 hr., Romberg = 70 min. Potential accuracy

Latitude	Longitude	18,284 Data points	256 × 256 Data points
0	0	1.8518	−0.2015
45	0	−0.3268	−0.0401
45	45	0.0703	0.0394
45	90	0.2382	−0.0391
45	135	0.2138	−0.0368
45	180	0.2856	−0.0383
45	225	0.1028	−0.0400
45	270	−0.4994	−0.0421
45	315	0.0289	−0.0411
90	0	−0.0250	0.0833
90	45	0.0377	0.0824
90	90	−0.0052	0.0825
90	135	0.4199	0.0870
90	180	0.1233	0.0863
90	225	0.1392	0.0852
90	270	0.0131	0.0845
90	315	0.1088	0.0828
135	0	−0.0321	−0.0413
135	45	−0.1243	−0.0415
135	90	−0.2636	−0.0411
135	135	−0.0836	−0.0392
135	180	0.1659	−0.0365
135	225	0.3877	−0.0362
135	270	0.2396	−0.0372
135	315	0.4246	−0.0394
180	0	1.0983	−0.2076

Tables 5 and 6 clearly show that the 256 × 256 model is superior to the least-squares model. In addition, it should be noted that the least-squares model required almost 50 times more computer time than the Romberg 256 × 256 model.

V. Summary and Conclusions

In summary, this paper has shown that the orthogonality properties of the spherical harmonic functions and the power of Romberg numerical integration can be exploited to accurately compute the geopotential coefficients. In addition, the Romberg numerical integration scheme was shown to be more efficient than the standard surface integral and least-squares techniques. No comparisons were made with numerical integration schemes that utilize the Colombo fast-Fourier algorithm, and hence, the relative accuracy and efficiency for higher-order models computed via the fast-Fourier algorithm remains an open question.

The procedure outlined here was theoretical since it required the use of a network of potentials and not a network of the radial components of gravity. However, this technique could be easily adapted to recover the geopotential coefficients from the radial components of gravity.

The only theoretical weakness found in the technique was the problem of aliasing, but aliasing is inherent in all the equal-spacing methods used to generate the geopotential coefficients. However, if a Romberg-like numerical integration technique based on unequal spacing could be developed, then this weakness would disappear.

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